# 9.1 BASIC COMBINATORICS

Learning Targets:

1. Solve counting problems using tree diagrams, lists, and/or the multiplication counting principle
2. Determine whether a situation is counted with permutations or combinations.
3. Solve counting problems with permutations.
4. Solve counting problems with combinations.
5. Solve counting problems with distinguishable permutations.

*The Multiplication Counting Principle*

If there are *n* "tasks" to complete and there are $c_1$ ways to complete the first task, $c_2$ ways to complete the second task after the first is completed, $c_3$ ways to complete the third task after the first two have been completed, …, and $c_n$ ways to complete the $n^{\text{th}}$ task after all the others have been completed, then the total number of ways to complete the *n* tasks is given by $c_1 \cdot c_2 \cdot c_3 \cdots c_n$

*Permutations*

A permutation is the number of possible <u>ordered</u> arrangements of the elements of a set. Let *n* = the total number of elements in the set and *r* = the number of elements being selected to be put in order.

- If $n > r$, we use the formula: ${}_nP_r = \dfrac{n!}{(n-r)!}$
- If $n = r$, then ${}_nP_r = {}_nP_n = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!}$. Since 0! = 1, ${}_nP_n = n!$
- When the elements of a set are repeated, we need to know the number of *distinguishable permutations* or we must divide out the permutations that have been counted more than once…

  We use the formula: $\dfrac{n!}{n_1! \cdot n_2! \cdot n_3! ...}$ where $n_1, n_2, n_3$, etc = the number of repeated elements

*Combinations*

A combination is the number of possible arrangements of the elements of a set where <u>the order of the elements does not matter</u>. Let *n* = the total number of elements in the set and *r* = the number of elements being selected to be put in order.

Use the formula: ${}_nC_r = \dfrac{n!}{(n-r)! \cdot r!}$ or use the notation: $\dbinom{n}{r} = \dfrac{n!}{(n-r)! \cdot r!}$

**9.2 THE BINOMIAL THEOREM**

Learning Targets:
1. Create Pascal's Triangle to at least six rows using patterns.
2. Use combinations to find the numbers in any row of Pascal's Triangle.
3. Expand binomial expressions.
4. Find a specific coefficient of a term in a binomial expansion without expanding the entire expression.

From the Exploration 9.2…

$$(a+b)^0 = 1$$
$$(a+b)^1 = 1a^1b^0 + 1a^0b^1$$
$$(a+b)^2 = 1a^2b^0 + 2a^1b^1 + 1a^0b^2$$
$$(a+b)^3 = 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3$$
$$(a+b)^4 = 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4$$
$$(a+b)^5 = 1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5$$

*Pascal's Triangle*

If we just look at the triangular array of coefficients in the binomial expansions above, we get the first 5 rows of the pattern known as Pascal's Triangle.

*Example 1:* Find the coefficients in the 6th row of Pascal's Triangle.

*Patterns of Binomial Expansions:* $(a+b)^n$

- The coefficients for each term in the expansion come from either Pascal's Triangle or the Binomial Coefficients: ${}_nC_r$ for $r = \{0,1,2,3,...n\}$.
- The first term is $a^nb^0$ or $a^n$. The last term is $a^0b^n$ or $b^n$.
- The exponents for *a* decrease by 1 for each term while the exponents for *b* increase by 1 for each term.
- The sum of the exponents in each term is *n*.
- The expansion has *n* + 1 terms.

*Example 2*: Use Pascal's Triangle to expand the expression $(2x+y)^4$.

**Binomial Theorem:**

$$(a+b)^n = {}_nC_0a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \ldots {}_nC_{n-1}ab^{n-1} + {}_nC_nb^n$$

**or**

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

*Example 3*: Use the Binomial Theorem to expand the expression $(x-3y)^5$.

*Example 4*: Find the coefficient for the term containing $y^7$ in the expansion of $(5x-2y)^9$ without actually finding the entire expansion.

## 9.3 PROBABILITY

Learning Targets:

1. Solve basic probability problems of equally likely outcomes.
2. Understand when to add probabilities and when to multiply probabilities ("or" vs "and").
3. Solve probability problems involving the words at least, exactly, at most, etc.
4. Solve probability problems involving complements.
5. Use Venn diagrams and Tree diagrams to model and solve probability problems.
6. Compute probabilities using permutations or combinations.
7. Compute binomial experiment probabilities.

***Definition of Probability of an Event:*** $P(E) = \dfrac{\text{\# of outcomes in the Event (E)}}{\text{\# of outcomes in Sample Space (S)}}$

- $0 \le P(E) \le 1$
- $P(\varnothing) = 0$
- Sum of probabilities in S = 1
- Remember the events must be equally likely to occur.

**Word problems with probability…**

**"and"** → $P(A \text{ and } B) = P(A) \cdot P(B)$

**"or"** → $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

**"at least one"** → use cases or the Complement Rule: $P(\text{at least } 1) = 1 - P(\text{none})$

*Example 1:* In Xavier's box of cars, there are 5 red cars, 8 blue cars, 4 yellow cars and 3 orange cars. Assume each car is equally likely to be selected and the first car is put back BEFORE the next car is selected. Answer the following questions.

a) What is the probability one car selected is one of Xavier's favorite colors: orange or red?

b) What is the probability that a blue car is selected first and a yellow car is selected second?

c) What is the probability that a car selected is NOT blue?

**Conditional Probability:** Probability under (given that) a certain condition has occurred. Some previous event has already happened and it has changed the probability of the next event.

*Example 2:* In a box of chocolates, there are 4 vanilla crèmes out of the 12 chocolates.

a) What is the probability that you randomly pick 2 vanillas from the box?

b) What is the probability that a vanilla crème is selected $1^{st}$ and a vanilla crème is NOT selected $2^{nd}$ ?

**Using Tree diagrams**

*Example 3:* The math team orders pizza after its last meet to celebrate the win. They order one medium pepperoni (8 pieces) and one large extra cheese (12 pieces). The team captain picks two pieces at random for the coach. Calculate the following probabilities.

a) P(both cheese) b) P(one slice of each)

c) P($1^{st}$ slice is pep | want 1 slice of each)

**Binomial Distributions**

***A binomial experiment has the following features:***

1. There are repeated situations, called trials.
2. There are only two possible outcomes, often called success and failure, for each trial.
3. The trials are independent.
4. Each trial has the same probability of success.
5. The experiment has a fixed number of trials.

***Binomial Probability:***

$P$ (exactly $k$ successes in n trials) = ${}_nC_k \cdot (P)^k \cdot (1-P)^{n-k}$

where the probability of a success = $P$ and the probability of a failure = $(1–P)$

*Example 4:* Suppose James makes 80% of his free throws. If he shoots 20 free throws, and if his chance of making each one is independent of the other shots, what is the probability that he makes

a) all 20?

b) exactly 18?

c) at least 18?

d) at least 2?

## 9.4 SEQUENCES AND SERIES

Learning Targets:

1. Write the terms of an explicitly defined sequence.
2. Write the terms of a recursively defined sequence.
3. Determine whether a sequence is arithmetic, geometric or neither.
4. Find (recursive and explicit) formulas for an arithmetic sequence.
5. Find (recursive and explicit) formulas for a geometric sequence.

Day 1

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6. Use summation notation.
7. Rewrite a series using summation notation.
8. Find the sum of an arithmetic series.
9. Find the sum of a geometric series (both finite and infinite).
10. Tell whether a geometric series is convergent or divergent and why.

Day 2

## DAY 1: SEQUENCES

*Let's start with some basic definitions…*

***Sequence:*** Any set of numbers in a specific order. Each element in the set is called a _______________. We use a ____________________ to identify which term we are talking about. For example, $a_1$ means the first term, $a_2$ means the _______________ term, etc.

- Two Types of Sequences:
  - Infinite Sequences: follow the same pattern forever.
  - Finite Sequences: follow the same pattern for a fixed number of terms.
- *Explicit Formula:* Gives the sequence as a function of the number of terms, *n*.
- *Recursive Formulas:* Gives the first term **and** a function for finding subsequent terms based on the previous term. Always think of "*n* – 1" as "the previous".

*Example 1:* Tell whether given formula is explicit or recursive. Then, find the next three terms of the sequence:

a) $a_n = 10 \bullet 2^n$

b) $a_1 = 57$
$a_n = a_{n-1} - 5 \text{ for } n \geq 2$

**<u>Arithmetic Sequences:</u>** Any sequence whose successive terms have a common difference, *d*.

- *Explicit Formula:* $a_n = a_1 + d(n-1)$
- *Recursive Formula:* $\begin{cases} a_1 = \# \\ a_n = a_{n-1} + d \quad \text{for } n \geq 2 \end{cases}$

*Example 2:* Given the arithmetic sequence –12, –1, 10, ....

a) Find a recursive formula.  b) Find $a_{41}$.  c) Find an explicit formula.

*Example 3:* What is the 21st term of the arithmetic sequence 11, 4, – 3, …?

*Example 4:* For the sequence below, write an explicit formula **and** find the 20th term. You do NOT have to find the middle terms! $\begin{cases} a_1 = 3 \\ a_n = a_{n-1} + 4 \text{ for } n > 1 \end{cases}$

<u>*Think about this…*</u>

- Are the following sequences arithmetic?
- Find $a_{10}$ for each of the sequences WITHOUT finding the middle terms.

1. 729, 243, 81…
2. 1, –4, 16, –64, …

**<u>Geometric Sequences</u>:** Any sequence whose successive terms have a common ratio, *r*.

- *Explicit Formula:* $a_n = a_1 \bullet r^{n-1}$
- *Recursive Formula:* $\begin{cases} a_1 = \# \\ a_n = a_{n-1} \bullet r \quad \text{for } n \geq 2 \end{cases}$

*Example 5:* Given the sequence $\begin{cases} t_1 = 4 \\ t_n = 6(t_{n-1}) \quad \text{for } n > 1 \end{cases}$

a) Find the first 4 terms of the sequence.

b) Write an explicit formula for the sequence.

*Example 6:* Write a recursive formula for the sequence 50, 25, 12.5, 6.25, . . .

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*<u>Warm Up for 9.4 day 2:</u>* Find the <u>sum</u> of the integers from 1 to 100 without a calculator. NO CHEATING!!!!

**DAY 2: SERIES**

***Series*****:** the sum of the terms in a sequence.

- **Summation or Sigma Notation:** Shorthand notation used to write "find the sum of". Most often used with an explicit formula.

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \ldots + x_n$$

- *Finite Series:* Series with a fixed number of terms from a finite sequence.

- *Infinite Series:* Series whose terms continue indefinitely; terms come from an infinite sequence.

*Example 7:* Let sequence $A = \{8, 10, 13, 14, 17, 20\}$. Find:

a) $a_3$ b) $a_5$ c) $\sum_{i=1}^{3} a_i$ d) $\sum_{i=2}^{6} a_i$

*Example 8:* Write an expression for the sum of the infinite series using summation notation.

a) $-7 - 1 + 5 + 11 + \ldots$ b) 5 – 15 + 45 – 135 + . . .

**Arithmetic Series:** The sum of the terms in a finite arithmetic sequence.

- If you KNOW the last term… $S_n = \frac{n}{2}(a_1 + a_n)$

- Notice you need ________________________________________ to use this formula.

*Example 9:* Find the sum of the first 32 terms in the arithmetic series: –12 – 6 + 0 . . .

*Example 10:* Find the sum of the arithmetic sequence: 117, 110, 103, …, 33

**Geometric Series:** The sum of the terms in a geometric sequence.

- Sum of a finite geometric series… $S_n = \dfrac{a_1(1-r^n)}{1-r}$
- Notice you need ________________________________________ to use this formula.

*Example 11:* Find the sum of the first 11 terms of the geometric sequence: $6, -3, \frac{3}{2}, -\frac{3}{4}, ...$

*Example 12:* Find the sum of the geometric series: 2 – 6 + 18 – 54 + . . . – 486

**Finding the Sum of *Infinite* Geometric Series**

While we can find the sum of a finite geometric or arithmetic series, we can only find a sum of an infinite geometric series … and then only when a specific condition is met.

**IF _________________, THEN YOU CAN FIND THE SUM OF AN INFINITE GEOMETRIC SERIES.**

If the sum can be found, it is found using the formula _______________.

If a sum is possible, we say the series ___________. If a sum is not possible, we say the series _____________.

*Example 13:* Determine whether the series converges. If it converges, give the sum.

a) $\sum_{j=1}^{\infty} 3\left(\frac{1}{4}\right)^{j-1}$

b) $\frac{1}{2} + \frac{3}{2} + \frac{9}{2} + \frac{27}{2} + ...$