1.2 FUNCTIONS AND THEIR PROPERTIES

Learning Targets for 1.2

1. Determine whether a set of numbers or a graph is a function
2. Find the domain of a function given an set of numbers, an equation, or a graph
3. Describe the type of discontinuity in a graph as removable or non-removable
4. For a given function, describe the intervals of increasing and decreasing
5. Label a graph as bounded above, bounded below, bounded, or unbounded.
6. Find all relative extrema on a graph
7. Understand the difference between absolute and relative extrema
8. Describe the symmetry of a graph as odd or even
9. Prove a function is odd, even, or neither
10. Given a rational equation, find vertical asymptotes and holes (if they exist)
11. Given a rational equation, find the end behavior model
12. Given a rational equation, describe the end behavior using end behavior and limit notation.
13. Given a rational equation, use the end behavior to find any horizontal asymptotes.
14. Given a rational equation, determine when slant (oblique) asymptotes exist and find them.

This section is full of vocabulary (obvious from the list above?). We will be investigating functions, and you will need to answer questions to determine how each of these properties are applied to various functions.

Function Definition and Notation

In the last chapter, we used the phrase "y is a function of x". But what is a function? In Algebra 1, we defined a function as a rule that assigned one and only one (a unique) output for every input. We called the input the *domain* and the output the *range*. Usually, the set of possible x-values is the domain, and the resulting set of possible y-values is the range.

Definition: Function

A **function** from a set D to a set R is a rule that assigns a unique element in R to each element in D.

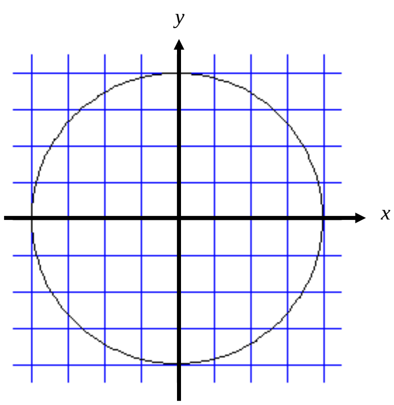
To determine whether or not a graph is a function, you can use the **vertical line test**. If any vertical line intersects a graph more than once, then that graph is NOT a function.

Example 1: A relation is ANY set of ordered pairs. State the domain and range of each relation, then tell whether or not the relation is a function.

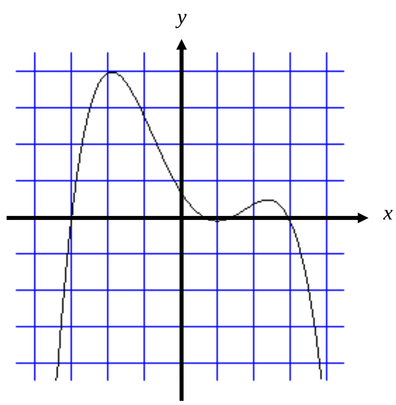
a) $\{(-3,0),(4,2),(2,-6)\}$

b) $\{(4,-2),(4,2),(9,-3)(-9,-3)\}$

c)



d)



For many functions, the domain is all real numbers, or $\mathbb{R}$. We typically start with this, and then see if there are any values of x that cannot be used.

The domain of a function can be restricted for 3 reasons that you need to be aware of in this course:

1. $\sqrt{\text{NO negatives}}$ … no negative numbers inside a square root
2. $\dfrac{**}{\text{NO zeros}}$ … no zeros in the denominator of a fraction
3. $\log(\text{NO negatives ... and NO zeros})$. … no negatives and no zeros inside a logarithm

Example 2: Without using a graphing calculator, what is the domain of each of the following functions?

a) $y = \sqrt{8-x}$

c) $y = \ln x$.

b) $h(x) = \dfrac{x^2-9}{x+3}$

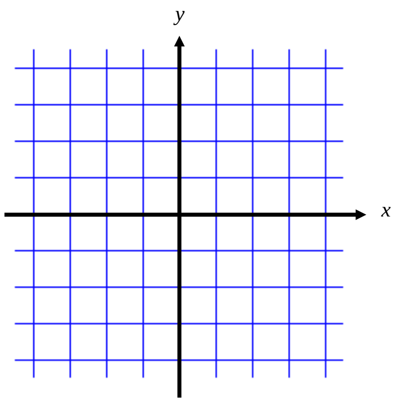
d) $y = \dfrac{\sqrt{x+2}}{x^2-x-12}$

<u>*Continuity*</u>

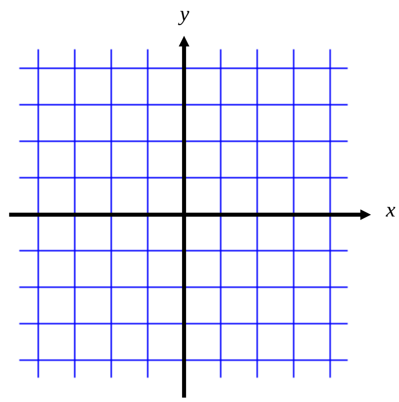
In non – technical terms, a function is continuous if you can draw the function "without ever lifting your pencil".

Example 3: Graph each of the following functions. What do you notice? What happens when $x = 2$ on the graph of b?

a) $f(x) = x+2$

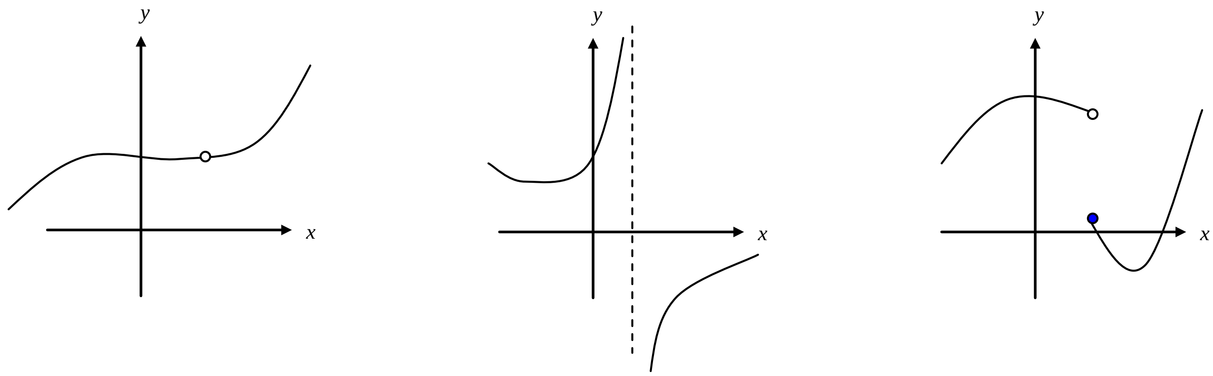


b) $g(x) = \dfrac{x^2-4}{x-2}$

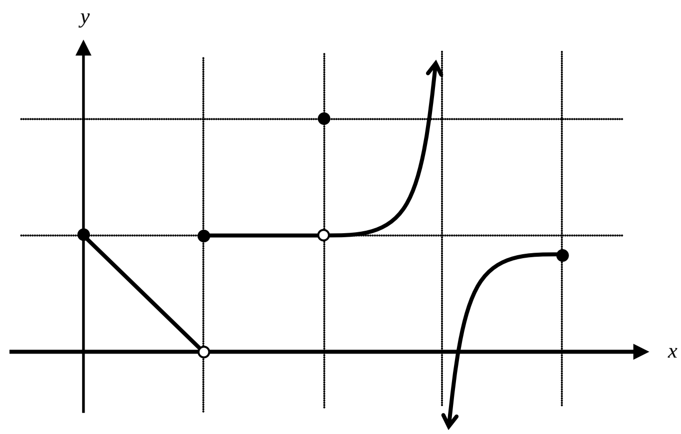


Example 4: What is the domain and range of the two functions above?

Example 5: The following graphs demonstrate three types of discontinuous graphs. Label each type as removable or non-removable.



Example 6: Tell where the function graphed below is discontinuous. Describe each discontinuity.



Example 7: What is the domain and range of the function above?

Example 8: Using your calculator, graph each of the following functions. Determine if it has discontinuity at $x = 0$. If there is a discontinuity, tell whether it is removable or non-removable.

a) $f(x) = x^3 - 2x^2 + 1$

b) $g(x) = \dfrac{9}{x}$

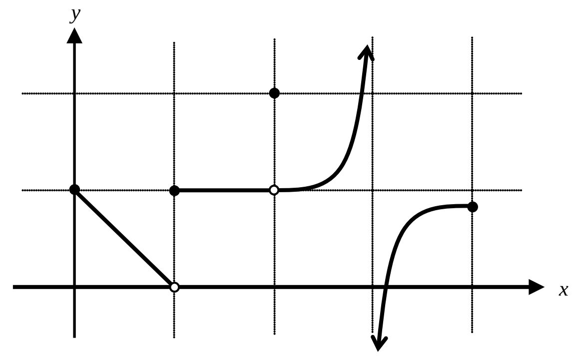
c) $h(x) = \dfrac{x}{x-3}$

d) $k(x) = \dfrac{x^2 - 2x}{x}$

Increasing/Decreasing Functions

While there are technical definitions for increasing and decreasing, just remember to read the graph LEFT to RIGHT.

Example 9: Using the graph below, what intervals is the function increasing? Decreasing? Constant?



Boundedness

You need to understand the difference between the following terms:

BOUNDED BELOW:

BOUNDED ABOVE:

BOUNDED:

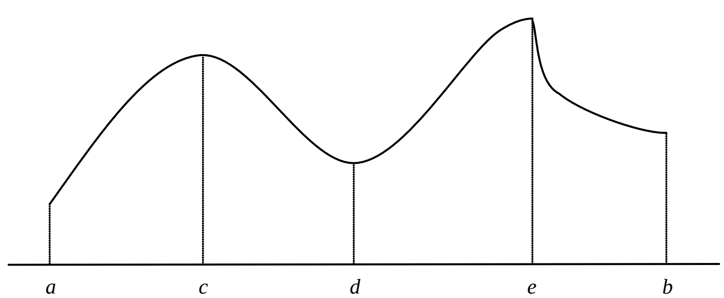
Local and Absolute Extrema

Extrema is the plural form of one extreme value. Extrema is one word that includes maximums and minimums.

Local Extrema are points bigger (or smaller) than ____.

Absolute Extrema are points bigger (or smaller) than ____.

Example: Suppose the following function is defined on the interval [a, b]. Identify the location and types of extrema.



Example 10: Use your graphing calculator, to graph the function $g(x) = -x^3 + 2x - 3$.

a) Identify all extrema.

b) Identify the intervals on which the function is increasing, decreasing, or constant.

Symmetry

The next topic we concern ourselves with when dealing with functions is the idea of symmetry. Symmetry on a graph means the functions "look the same" on one side as it does on another. We are most concerned with the types of symmetry that can be explored numerically and algebraically in terms of ODD and EVEN functions.

3 Types of Symmetry

Symmetry with respect to the *y*-axis: **EVEN FUNCTIONS**

Graphically Numerically Algebraically

Symmetry with respect to the origin: **ODD FUNCTIONS**

Graphically Numerically Algebraically

Symmetry with respect to the *x*-axis: **NOT A FUNCTION**

Graphically Numerically Algebraically

Odd vs Even Functions

While there are many functions out there that are neither even nor odd, your concern with odd and even functions is twofold …

#1: Identify graphs of functions that are Odd or Even

#2: PROVE a function is Odd or Even

While the first item above can be done graphically, numerically, or algebraically, the second is done ONLY algebraically.

Example 11: Prove each function is odd, even, or neither.

a) $f(x) = 5x^2 + 5$ b) $g(x) = x^2 + 8x + 12$ c) $h(x) = -4x^3 + 2x$

Vertical Asymptotes

Example 12: Graph the function $f(x) = \frac{1}{x-1}$. What happens at x = 1? … Why?

To describe a vertical asymptote, we must talk about what happens to the *y*-values of a function as the *x*-values get "close" to a certain number.

We use the notation $x \to 1$ to say ____.

If we only want x to approach 1 from the right side, we use the notation ________________________.

If we only want x to approach 1 from the left side, we use the notation ________________________.

♬: In calculus, we use what is called "limit notation" … $\lim_{x \to c^+} f(x) = \pm\infty$ or $\lim_{x \to c^-} f(x) = \pm\infty$.

Vertical asymptotes occur in rational functions when ____.

If you plug a point into your function and get _________, you should look for a removable discontinuity … a.k.a. "a hole".

Horizontal Asymptotes and End Behavior

Horizontal Asymptotes occur when the function values (*y*-values) get "close" to a specific number as the *x*-values get really, really large in the positive direction (∞) or negative direction ($-\infty$).

As $x \to \pm\infty$, we say the *end behavior* of the function is a description of what $f(x)$ approaches.

IF $f(x) \to$ "a number" as $x \to \pm\infty$, we say the function has a *horizontal asymptote*.

Example 13: Graph the function $g\left(x\right)=\dfrac{4x^3}{27-x^3}$. Where does the horizontal asymptote occur?

The end behavior model of a polynomial is the leading coefficient and the highest power of the variable.

A rational function is just two polynomials divided, so the end behavior model of a rational function is just the end behavior model of the numerator divided by the end behavior model of the denominator.

Example 14: What is the end behavior model for $g\left(x\right)=\dfrac{4x^3}{27-x^3}$? How is this related to the horizontal asymptote.

Example 15: Consider the function $h\left(x\right)=\dfrac{1}{x}$. As $x\to\infty$, where does h (x) approach?

<u>*Summary for Horizontal Asymptotes*</u>

For rational functions we have the following results.

If $f\left(x\right)=ax^m+\cdots$ and $g\left(x\right)=bx^n+\cdots$ then $\dfrac{f\left(x\right)}{g\left(x\right)}$ takes on three different forms.

	End Behavior Model	End Behavior	Asymptote
$m=n$			
$m<n$			
$m>n$			

Example 16: Find the end behavior models of each function and determine what (if any) the horizontal asymptotes are.

a) $f\left(x\right)=\dfrac{5x^5-7x^3+2x-8}{10x^5+16x^4-3x^2+9}$ b) $g\left(x\right)=\dfrac{5x^8-7x^3+2x-8}{10x^5+16x^4-3x^2+9}$ c) $h\left(x\right)=\dfrac{5x^4-7x^3+2x-8}{10x^6+16x^4-3x^2+9}$

Example 17: Go back and find the slanted (or oblique) asymptote for the graph in part *b* above.

Example 18: Find ALL vertical and horizontal asymptotes for the following functions.

a) $f(x)=\dfrac{2x-1}{3x+5}$

b) $g(x)=\dfrac{(3x-5)(x-8)}{x^2-4}$

c) $h(x)=\dfrac{2x-9}{x^2-x-6}$

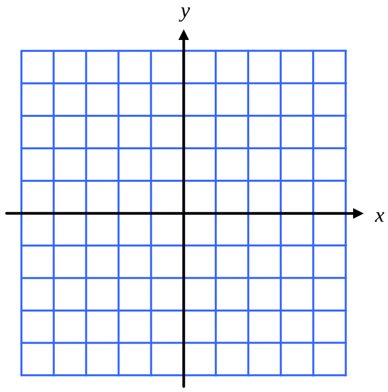
d) $k(x)=\dfrac{2x}{x^2-x-6}$

1.3 TWELVE BASIC FUNCTIONS

Lesson Targets for 1.3
1. Graph and Identify all 12 parent functions
2. Graph a piecewise function

Parent Function #1: Linear Function (book refers to this as the identity function): Equation: ____________

Graph this function (label 5 points)



Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Increasing/Decreasing:

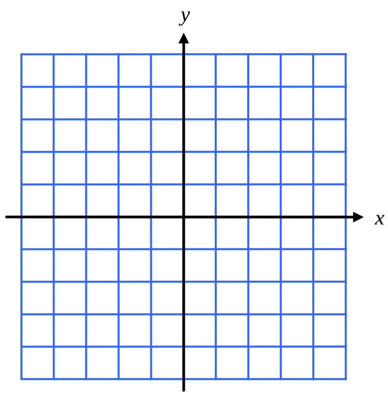
Extrema:

Domain:

Range:

Parent Function #2: Quadratic Function (book refers to this as the squaring function): Equation: _____________

Graph this function (label 5 points)



Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Increasing/Decreasing:

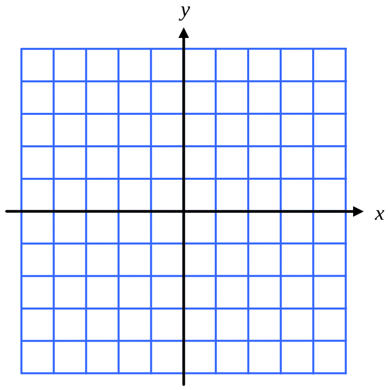
Extrema:

Domain:

Range:

Parent Function #3: Cubic Function (the book refers to this as the cubing function) : Equation: _____________

Graph this function (label 3 points)



Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Increasing/Decreasing:

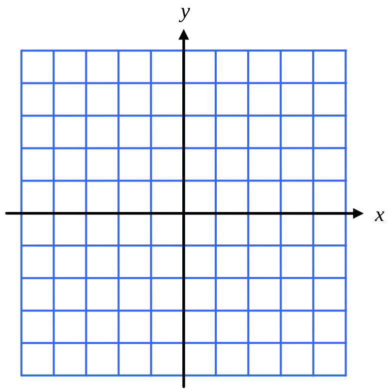
Extrema:

Domain:

Range:

Parent Function #4: Inverse Linear Function (book refers to this as the Reciprocal Function): Equation: _____________

Graph this function
(label 2 points, a H.A., and a V.A.)



Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Increasing/Decreasing:

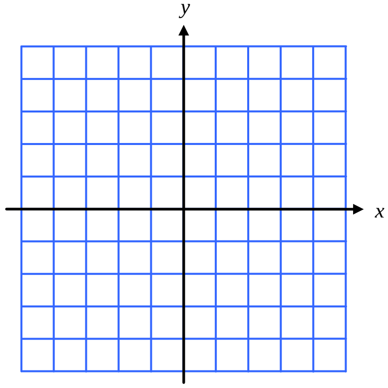
Extrema:

Domain:

Range:

Parent Function #5: Square Root Function: Equation: ____________

Graph this function (label 3 points)



Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Increasing/Decreasing:

Domain:

Extrema:

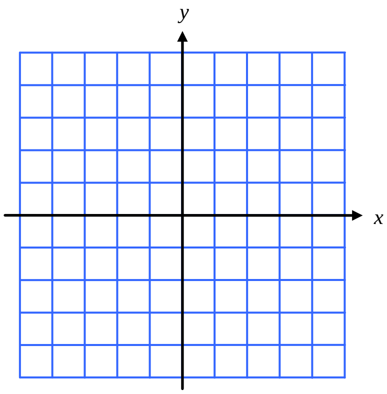
Range:

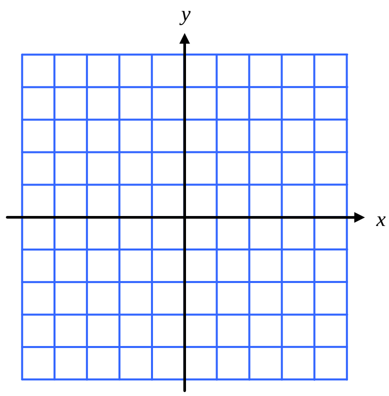
Parent Function #6: Exponential Function: (the book uses only $f(x) = e^x$)

I would like you to use the equation $f(x) = b^x$,

where $b > 1$ represents ____________, and $0 < b < 1$ represents _______________.

Graph each function (label 2 points and a H.A.)





Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Increasing/Decreasing:

Domain:

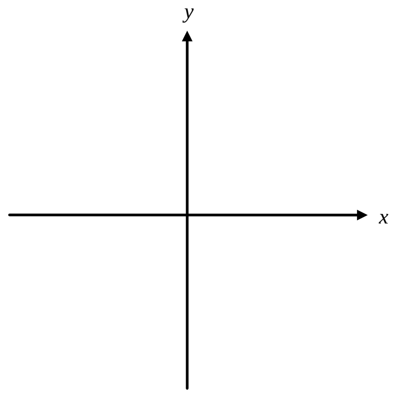
Range:

Extrema:

Parent Function #7: Logarithm Function: (the book only uses a natural logarithm)

I would like you to use $f(x) = \log_b x$

Graph this function (label 2 points and a V.A.)



Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Increasing/Decreasing:

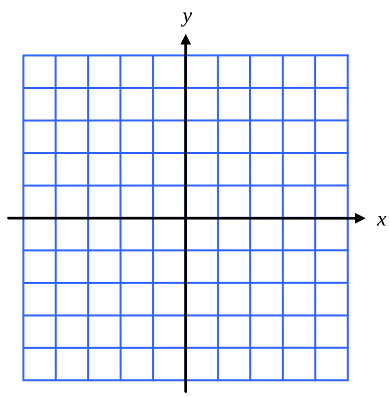
Extrema:

Domain:

Range:

Parent Function #8: Absolute Value Function: Equation: ______________

Graph this function (label 5 points)



Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Increasing/Decreasing:

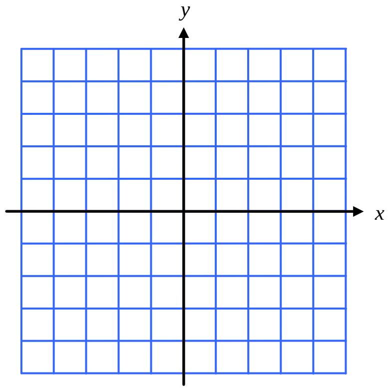
Extrema:

Domain:

Range:

Parent Function #9: Greatest Integer Function: Equation: ____________

Graph this function (label at least 6 points)



Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Increasing/Decreasing:

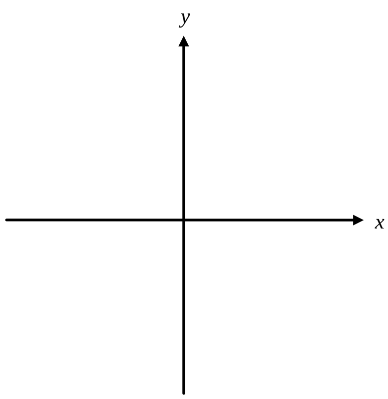
Extrema:

Domain:

Range:

Parent Function #10: Logistic Function: Equation: _____________

Graph this function (label 1 point and 2 H.A.)



Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Increasing/Decreasing:

Extrema:

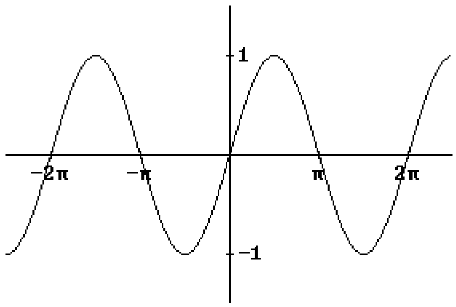
Domain:

Range:

You won't need to graph the next two until 2nd Semester … just be able to tell the difference between the two for now.

Parent Function #11: Sine Function: Equation: $y = \sin(x)$

Here's a graph of this function



Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

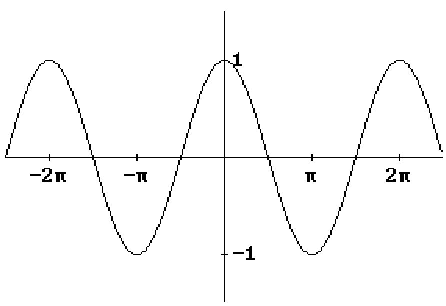
Domain:

Extrema:

Range:

Parent Function #11: Cosine Function: Equation: $y = \cos(x)$

Here's a graph of this function



Symmetry:

Boundedness:

Asymptotes:

Discontinuities:

Extrema:

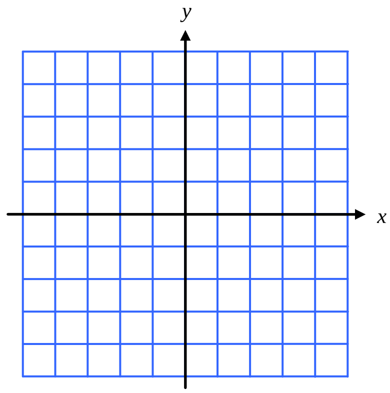
Domain:

Range:

Piecewise Functions

Piecewise Functions are functions that are defined "in pieces". Each piece is a portion of a graph with a limited domain.

Example 1: Graph the following piecewise function: $f(x)=\begin{cases} 3^x & ; x>0 \\ |x| & ; x\le 0 \end{cases}$



1.5 GRAPHICAL TRANSFORMATIONS

Learning Targets for 1.5
1. Be able to graph transformations of parent functions 2. Be able to adjust the function when there is a horizontal stretch/shrink AND a left/right movement.

You MUST be able to graph the parent functions from 1.3 in order to successful transform them.

A transformation is a stretch/shrink, a reflection, or simple movement of a parent function horizontally or vertically.

The general rule … "Inside … x's … opposite" and "Outside … y's … same"

Example 1: Suppose you are given the function $f(x)$. If a, b, c, and d are real numbers, our transformed function is

$$a \cdot f\left(b(x+c)\right)+d$$

a) Identify which letters are inside the function and which letters are outside.

b) For the outside letters, describe how they transform the original function.

c) For the inside letters, describe how they transform the original function.

Another way to look at this is with the following chart:

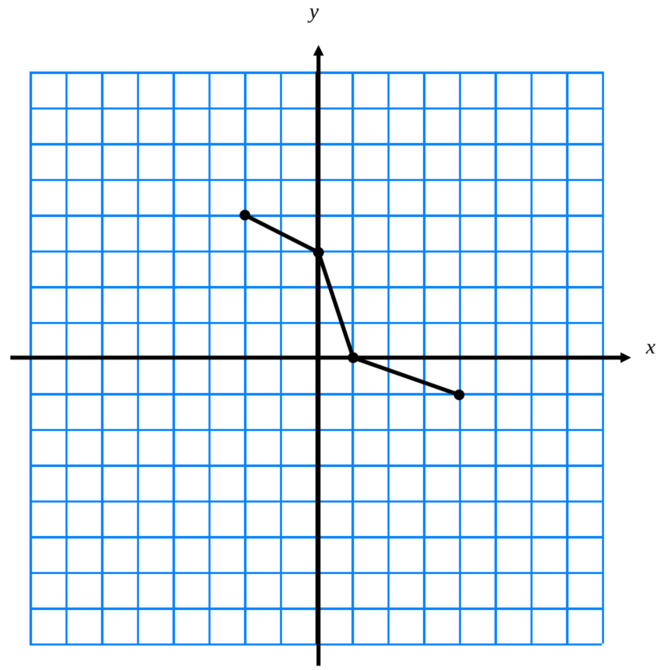
	Inside		Outside	
$+/-$	$f(x+c)$	$f(x-c)$	$f(x)+d$	$f(x)-d$
$\times/\div$	$f(bx)$	$f(-x)$	$a \cdot f(x)$	$-f(x)$

Example 2: Perform the following transformations on the graph of $f(x)$ below:

a) $f\left(x+3\right)+2$

b) $-2f\left(\dfrac{x}{3}\right)$

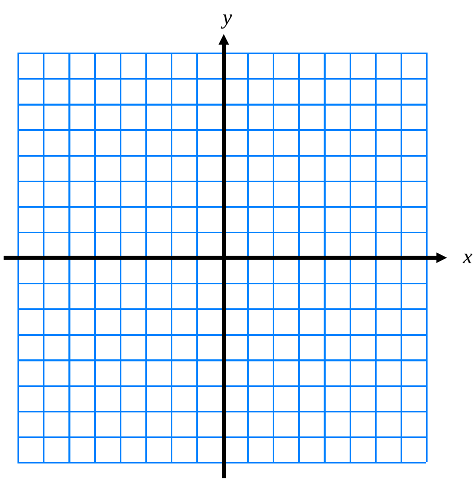
c) $f\left(2x+6\right)$

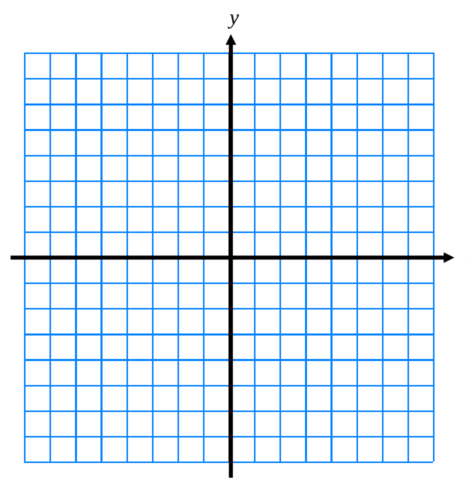


Example 3: Graph the following functions by transforming their parent functions: **MOST OFTEN MISSED**

a) $f\left(x\right)=\sqrt{3x-6}$

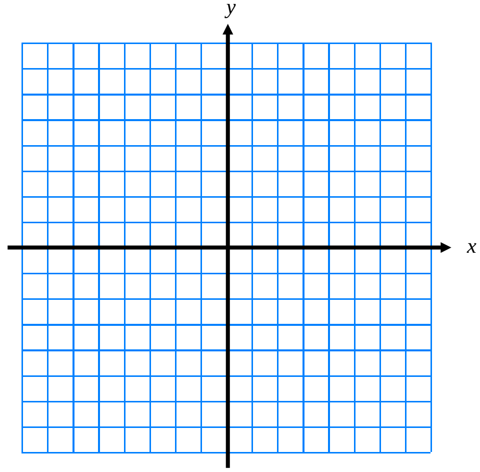
b) $g\left(x\right)=\sqrt{5-x}$



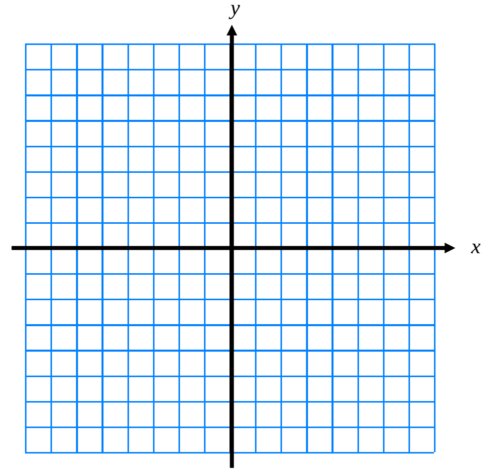


Example 4: Graph the following functions:

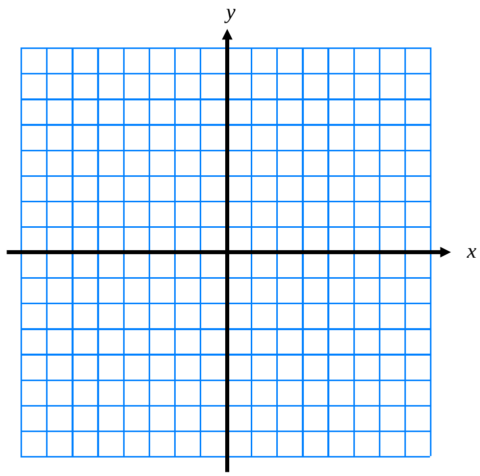
a) $h(x)=\dfrac{1}{x+2}-3$



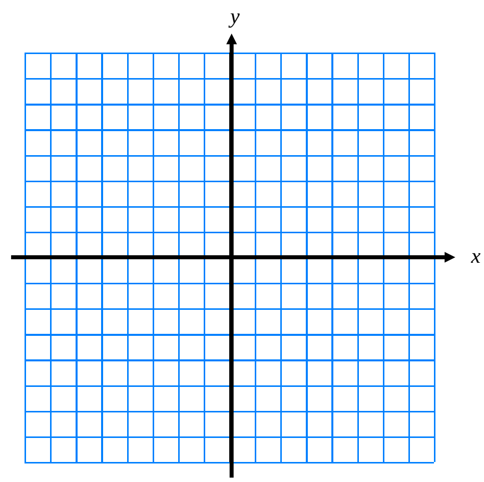
b) $k(x)=\log_4(x-2)+1$



c) $n(x)=3|x-2|+1$



d) $p(x)=3^{x/2}-1$



1.4 BUILDING FUNCTIONS FROM FUNCTIONS

Learning Target for 1.4

1. Be able to Add, Subtract, Multiply, and Divide two functions
2. Be able to Compose two functions
3. Find an Inverse Functions Graphically, Numerically, and Algebraically
4. Prove two functions are Inverses

Creating functions from other functions can be done in a variety of ways. We are going to add, subtract, multiply, divide, and compose functions to create new functions.

Example 1: Consider the two functions $f(x) = 6 - x$ and $g(x) = \sqrt{x+2}$. What is the Domain of each?

a) Find $(f+g)(x)$. State the Domain of the new function.

b) Find $(f-g)(x)$. State the Domain of the new function.

c) Find $(fg)(x)$. State the Domain of the new function.

d) Find $\left(\frac{f}{g}\right)(x)$. State the Domain of the new function.

Composite Functions

When the range of one function is used as the domain of a second function we call the entire function a composite function.

We use the notation $(f \circ g)(x) = f(g(x))$ to describe composite functions.

This is read as "*f* composed with *g*" or "*f* of *g* of *x*".

The function *f* in the example above is an example of a composite function. The linear function "$x + 2$" is applied first, then the square root function.

Example 2: Suppose $f(x) = 1 - x^2$ and $g(x) = \sqrt{x}$.

a) Find $g(f(x))$. What is the domain of $g(f(x))$?

b) Find $f(g(x))$. What is the domain of $g(f(x))$?

Example 3: Suppose $f(x)=\sqrt{x-5}$, and $g(x)=x^2+5$.

a) Find $f(g(x))$.

b) Find $g(f(x))$.

Inverse Functions

A function has an inverse function if and only if the original function passes the Horizontal Line Test. The Horizontal Line Test works just like the Vertical Line Test (it's just horizontal ☺). All an inverse function does is switch the *x* and *y* or the domain and range.

Example: Does $y=x^2+5x$ have an inverse? Why or why not?

Example: Does $y=x^3+x$ have an inverse? Why or why not?

Once we know whether a function has an inverse, our next task is to find an equation and/or a graph for the inverse.

Finding the Inverse Graphically

Reflect the graph of the original function over the line $y = x$.

Finding the Inverse Numerically

Plot the reverse of the coordinates.

Finding the Inverse Algebraically

Switch the *x* and *y* in the original equation, then solve the new equation for *y* in order to write *y* as a function of *x*. Your NEW *y* can be written as $f^{-1}(x)$

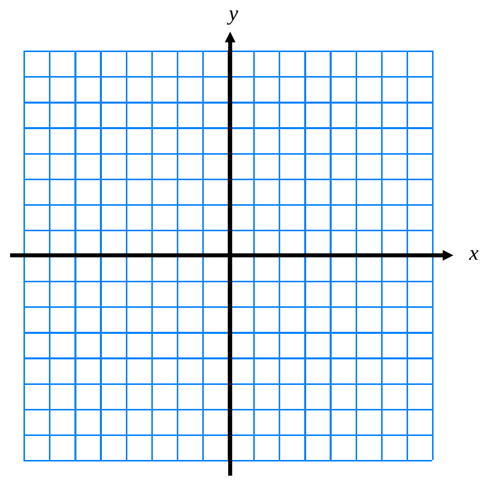
Example 4: Let $f(x)=x^3-1$.

a) Graph the function on the grid to the right.

b) Draw the line $y = x$

c) Reflect the graph of $f(x)$ over the line $y = x$.

d) Find the inverse of the function algebraically.



Verifying Inverses

It is one thing to find the inverse function (either graphically or algebraically), but it is another to verify that two functions are actually inverses. Whenever you are verifying anything in mathematics, you must go back and use the definition.

Definition: Inverse Function

A function $f(x)$ has an inverse $f^{-1}(x)$ if and only if $f\left(f^{-1}(x)\right)=x=f^{-1}\left(f(x)\right)$

Example 5: According to this definition, how many composite functions must you use to check whether or not two functions are inverses of each other?

Example 6: **Find** $f^{-1}(x)$ and **verify** if $f(x)=3x-2$.

Example 7: Find $f^{-1}(x)$ and verify if $f(x)=\dfrac{x+3}{x-2}$.

1.6 MODELING WITH FUNCTIONS

Lesson Targets for 1.6

1. Change English statements into mathematical expressions
2. Write equations to model given situations.
3. Use equations to solve percentage and mixture problems.

Example 1: Write a mathematical expression for the quantity described verbally.
(An expression has no equal sign, and can therefore NOT be solved.)

a) One less than five times a number x.

b) A number x decreased by six and then doubled.

c) A salary after a 4.4% increase, if the original salary is x dollars.

Example 2: Write an equation for each of the following statements:

a) One leg of a right triangle is three times as long as the other. Write the length of the hypotenuse as a function of the length of the shorter leg.

b) The height of a right circular cylinder equals its diameter. Write the volume of the cylinder as a function of its radius.

Example 3: For each statement below, do the following:

1. Write an equation (be sure to define any variables used).
2. Solve the equation, and answer the question.

a) One positive number is twice another positive number. The sum of the two numbers is 390. Find the two numbers.

b) Joe Pearlman received a 3.5% pay decrease. His salary after the decrease was $27,985. What was his salary before the decrease?

c) Investment returns Jackie invests $25,000 part at 5.5% annual interest and the balance at 8.3% annual interest. How much is invested at each rate if Jackie receives a 1-year interest payment of $1571?

d) The chemistry lab at the University of Hardwoods keeps two acid solutions on hand. One is 20% acid and the other is 35% acid. How much 20% solution and how much 35% acid solution should be used to prepare 25 liters of a 26% acid solution?