Calculus
Chapter 3.1 – 3.5 Review Sheet Solutions

1. f is differentiable on the interval (–4, 4) except at $x = -2$ (corner), $x = -1/2$ (vertical tangent line), $x = 1$ (corner), $x = 2$ (corner), $x = 3$ (discontinuity)

2. To find marginal profit, find the derivative of P … $P'(x) = 12x^2 - 7$

The marginal profit of the 12th item (currently producing 11 items) is $P'(11) = 12(11)^2 - 7 = 1445$

3. A derivative at 2 exists when the derivative from the left and right sides are the same. Also, keep in mind that if the function is NOT continuous at 2, $f'(2)$ will not exist regardless of the left and right derivatives. (see notecards for definition of a derivative at $x = c$ and at an endpoint)

4. First find the derivative of the function: $f'(x) = \begin{cases} 4ax & x > 1 \\ -3 & x < 1 \end{cases}$. In order for the derivative to exist at $x = 1$, the derivative from the left and the derivative from the right must be equal. So, $4a(1) = -3$, meaning $a = \frac{-3}{4}$.

In order for f to be continuous at $x = 1$, $\lim_{x \to 1} f(x) = f(1)$. In order for this limit to exist, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$.
So, evaluating the left and right hand limits and substituting what we now know a equals we get

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^+} f(x) \\ -3(1) + 4 &= 2a(1)^2 + b \\ 1 &= 2\left(\tfrac{-3}{4}\right) + b \\ \tfrac{5}{2} &= b \end{aligned}$$

Therefore, to be continuous and differentiable, $a = \frac{-3}{4}$ and $b = \frac{5}{2}$.

5. a) Average velocity is $\frac{\text{total distance}}{\text{total time}} = \frac{x(3) - x(0)}{3 - 0} = \frac{-3 - 12}{3} = -5$ ft/s

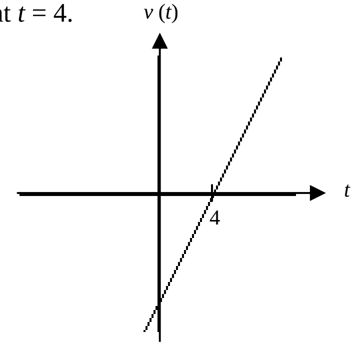
b) Velocity is the derivative of position: $v(t) = 2t - 8$ … $v(4) = 0$ ft/s

c) Since velocity is 0 at $t = 4$, the object is stopped at $t = 4$.

d) Acceleration is the derivative of velocity: $a(t) = 2\ ^{ft}/_{s^2}$.

e) The object changes direction when the velocity changes sign. There are two ways to look at this ….

Graphically: Notice at $t = 4$, the graph of the velocity function changes from negative to positive, so the object changes direction at $t = 4$.



Numerically: Since we already know the velocity is 0 at $t = 4$, then all we need to do is check the sign of the velocity function for values of t less than 4 and again for values of t greater than 4. Since $v(3) = -2$ and $v(5) = 2$, the object changes direction at $t = 4$.

f) The object slows down when the velocity and acceleration have opposite signs. Since the acceleration is always positive, then the object is slowing down when velocity is negative. Thus, when $t < 4$.

g) The object is moving left when velocity is negative. Again, when $t < 4$.

6. The equation of a tangent line requires a point and a slope. To get the point, plug $x = \frac{\pi}{2}$ into the function. So, the point is $\left(\frac{\pi}{2}, 0\right)$. To find the slope, take a derivative. Using the PRODUCT RULE (let $u = 2\sin x$ and $v = \cos x$) we have $y' = 2\sin x(-\sin x) + \cos x(2\cos x) = -2\sin^2 x + 2\cos^2 x$. When $x = \frac{\pi}{2}$, $y' = -2 - 0 = -2$. So, the equation of the tangent line is $y - 0 = -2\left(x - \frac{\pi}{2}\right)$ or $y = -2x + \pi$. … Try graphing both the function and the tangent line on your calculator as a way to check.

7. The first limit is the limit definition of the derivative of $\cos x$ at $x = \frac{\pi}{2}$. Since the derivative of $\cos x$ is $-\sin x$, then the answer is $-\sin\left(\frac{\pi}{2}\right) = -1$.
The second limit is the limit definition of the derivative of $\sqrt{x}$ at $x = 4$. Since the derivative of $\sqrt{x}$ is $\frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$, then the answer is $\frac{1}{2\sqrt{4}} = \frac{1}{4}$.

8. Using the alternative definition we get

$$\begin{aligned} f'(2) &= \lim_{x\to 2}\frac{f(x) - f(2)}{x-2} \\ &= \lim_{x\to 2}\frac{\left(\frac{1}{x} - \frac{1}{2}\right)}{x-2} \\ &= \lim_{x\to 2}\frac{\left(\frac{2-x}{2x}\right)}{x-2} \\ &= \lim_{x\to 2}\frac{2-x}{2x}\cdot\frac{1}{x-2} \\ &= \lim_{x\to 2}\frac{-1}{2x} \\ &= \frac{-1}{4} \end{aligned}$$

You should be able to <u>check</u> this using the power rule … since $f(x) = \frac{1}{x} = x^{-1}$, then $f'(x) = -1x^{-2} = \frac{-1}{x^2}$. Therefore, $f'(2) = \frac{-1}{4}$.

9. Using the alternative definition we get

$$\begin{aligned} f'(1) &= \lim_{x\to 1}\frac{f(x) - f(1)}{x-1} \\ &= \lim_{x\to 1}\frac{(3x^2 + 5x) - (8)}{x-1} \\ &= \lim_{x\to 1}\frac{3x^2 + 5x - 8}{x-1} \\ &= \lim_{x\to 1}\frac{(3x+8)(x-1)}{x-1} \\ &= \lim_{x\to 1}(3x+8) \\ &= 11 \end{aligned}$$

You should be able to <u>check</u> this using the power rule … since $f(x) = 3x^2 + 5x$, then $f'(x) = 6x + 5$. Therefore, $f'(1) = 6(1) + 5 = 11$.

10. $f'(3)$ is the derivative at $x = 3$ … or the rate of change in $/min at 3 minutes. To approximate this value, use the slope of a secant line (any secant line that includes $x = 3$) … ANY of the following responses would be considered acceptable, as long as you explain what you are doing …

$$\frac{f(4)-f(3)}{4-3}=\frac{11-9}{1}=2$$
$$\frac{f(3)-f(2)}{3-2}=\frac{9-6}{1}=3$$
$$\frac{f(4)-f(2)}{4-2}=\frac{11-6}{2}=\frac{5}{2}$$

11. The first worksheet left off the domain … you should limit the domain of t to be $0 \le t \le 2\pi$ for this entire problem!

a) Velocity is the derivative of position (use product rule) … $v(x)=s'(x)=x^2(\cos x)+\sin x(2x)$

b) The acceleration is the derivative of the velocity function (or second derivative of position) … use the product rule for each part … $a(x)=s''(x)$

$$= x^2(-\sin x)+\cos x(2x)+\sin x(2)+2x(\cos x)$$
$$= -x^2 \sin x + 4x\cos x + 2\sin x$$

c) The object is stopped when velocity equal 0 … using a calculator the object is stopped at $x = 0$, $x \approx 2.289$, and $x \approx 5.087$.

d) The particle changes direction when the velocity changes sign … once again, using the graph, the velocity changes sign when $x \approx 2.289$ and $x \approx 5.087$.

e) The particle is speeding up when the velocity and acceleration have the same sign … using your calculator, graph the acceleration function and find the zeros so that you can determine when the acceleration changes sign … The zeros of the acceleration function are $x = 0$, $x \approx 1.520$, and $x \approx 3.994$. Since velocity and acceleration are both positive on $(0,1.520)$ and $(5.087, 2\pi)$ and they are both negative on $(2.289, 3.994)$ these are the three intervals for which the particle is speeding up

… Another way to do this is to consider that speed is the absolute value of velocity … so graph the speed function and find the intervals where the speed is increasing (going up). … to do this you would need to find the zeros and the maximums, but you would end up with the same intervals.

f) Zeros of $s(x)$ would be the same as the zeros of $\sin x$ … $x = 0, \pi, 2\pi$

g) Zeros of $v(x)$ are $x = 0$, $x \approx 2.289$, and $x \approx 5.087$.

h) Zeros of $a(x)$ are $x = 0$, $x \approx 1.520$, and $x \approx 3.994$.

12. If $f(x)$ has a derivative at $x = 2$, then it must also be continuous at $x = 2$… so *a* MUST BE TRUE (part of the definition of continuity), *b* MUST BE TRUE (because the $f'(2)$ is notation for the derivative at $x = 2$), *c* is not necessarily true because we have been given nothing about the second derivative, *d* MUST BE TRUE (because if a function has a derivative at a point, then it must be continuous at that point), *e* MUST BE TRUE (because that is the alternative definition of the derivative at $x = 2$), and *f* MUST BE TRUE (because that is the standard definition of the derivative at $x = 2$).