2.1 RATES OF CHANGE AND LIMITS

Limits

Limits are what separate Calculus from pre – calculus. Using a limit is also the foundational principle behind the two most important concepts in calculus, derivatives and integrals. Limits can be found using substitution, graphical investigation, numerical approximation, algebra, or some combination of these.

Average and Instantaneous Velocity

In pre – calculus courses, you used the formula $d = rt$ to determine the speed of an object. What you found was the object's average speed. A moving body's **average speed** during an interval of time is found by dividing the total distance covered by the elapsed time. (Speed is always positive … Velocity indicates direction and can be negative.) We are going to find the **average velocity**.

If an object is dropped from an initial height of h_0, we can use the position function $s(t) = -16t^2 + h_0$ to model the height, s, (in feet) of an object that has fallen for t seconds.

Example: Wile E. Coyote, once again trying to catch the Road Runner, waits for the nastily speedy bird atop a 900 foot cliff. With his Acme Rocket Pac strapped to his back, Wile E. is poised to leap from the cliff, fire up his rocket pack, and finally partake of a juicy road runner roast. Seconds later, the Road Runner zips by and Wile E. leaps from the cliff. Alas, as always, the rocket malfunctions and fails to fire, sending poor Wile E. plummeting to the road below disappearing into a cloud of dust.

    a) What is the position function for Wile E. Coyote?

    b) Find Wile E.'s average velocity for the first 3 seconds.

    c) Find Wile E.'s average velocity between $t = 2$ and $t = 3$ seconds.

    d) Find Wile E.'s velocity at the instant $t = 3$ seconds.

The problem with the part c is that we are trying to find the *instantaneous velocity*. Without the concept of a limit, we could not find the answer to part *c*. Using a limit to solve this problem involves studying what happens as we get "close" to 3 seconds.

*Example*: Find the average velocity between $t = 2.5$ and $t = 3$ seconds.

Example: Find the average velocity between $t = 2.9$ and $t = 3$ seconds.

Example: Find the average velocity between $t = 2.99$ and $t = 3$ seconds.

Example: Find the average velocity between $t = 2.999$ and $t = 3$ seconds.

So, even though we cannot find the average velocity at exactly $t = 3$ seconds, we can discover what Wile E.'s velocity is approaching at $t = 3$ seconds.

Example: Sketch the graph of $f(x) = \dfrac{x^2 - 4}{x - 2}$; $x \neq 2$.
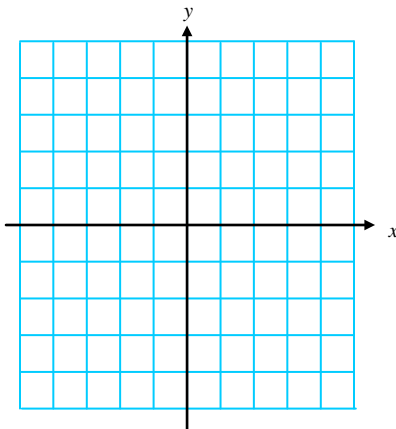
a) What happens at $x = 2$?

b) Complete the table of values below to determine what happens as x gets "close" to 2.

x approaches 2 from the **left** ⟶ | ⟵ *x* approaches 2 from the **right**

x	1.5	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25	2.5
$f(x)$											

Informal Definition of a Limit

Suppose a function f is defined on an interval around $x = c$, but possibly not at the point $x = c$ itself. Suppose that as x becomes sufficiently close to c, $f(x)$ becomes as close to a single number L as we please.
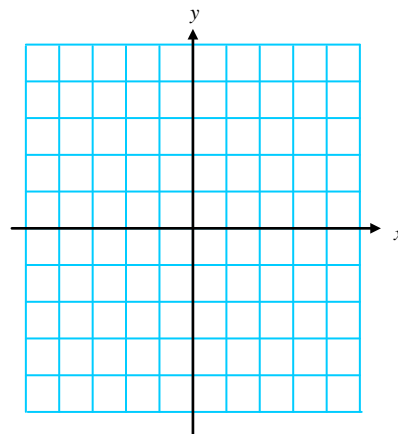We then say that the **limit of $f(x)$ as x approaches c is L**, and we write
$$\lim_{x \to c} f(x) = L.$$

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c) Apply this definition to the function from above to find the $\lim\limits_{x \to 2} f(x)$.

Example: Use the graph to find $\lim\limits_{x \to 2} g(x)$, where g is defined as

$$g(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

When we say "$f(x)$ becomes as close to L as we please" in the informal definition, we mean that we can specify a maximum distance between $f(x)$ and L. This distance is given by

$$\left|f(x)-L\right| = \text{Distance between } f(x) \text{ and } L.$$

We use the Greek letter ε (epsilon) to stand for the maximum distance, so we require

$$\left|f(x)-L\right| < \varepsilon .$$

Similarly, we interpret "x becomes sufficiently close to c" to mean

$$\left|x-c\right| < \delta ,$$

where the Greek letter δ (delta) tells us how close x must be to c. Then

$$\lim_{x \to c} f(x) = L$$

means that we can make the distance $\left|f(x)-L\right|$ between the function values and L as small as we like (less than any number $\varepsilon > 0$) by making the distance $\left|x-c\right|$ between x and c sufficiently small (less than some $\delta > 0$).

Formal Definition of a Limit

Let c and L be real numbers. The function f **has a limit L as x approaches c** if, given any positive number ε, there is a positive number δ such that for all x,

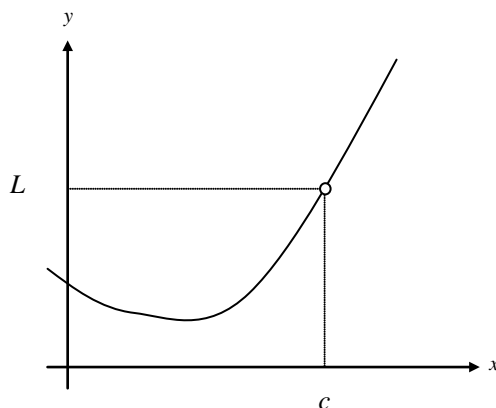$$0 < \left|x-c\right| < \delta \Rightarrow \left|f(x)-L\right| < \varepsilon .$$

We write

$$\lim_{x \to c} f(x) = L$$

(Just for fun ☺) Symbolically this can be written as follows:

$$\lim_{x \to c} f(x) = L \Leftrightarrow (\forall \varepsilon > 0)(\exists \delta > 0) \ni \left(0 < \left|x-c\right| < \delta\right) \Rightarrow \left(\left|f(x)-L\right| < \varepsilon\right)$$

Example: Consider the following function. Graphically show the definition of a limit.


When Limits Do Not Exist

If there does not exist a <u>number</u> L satisfying the condition in the definition, then we say the $\lim_{x \to c} f(x)$ does not exist.

Limits typically fail for three reasons:
1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches c.
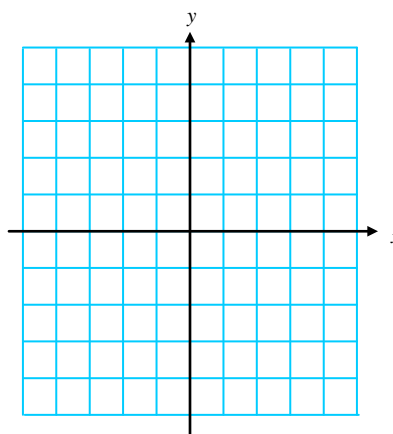3. $f(x)$ oscillates between two fixed values as x approaches c.

Example: Investigate (use a graph and/or table) the existence of the following limits.

(a) $\lim\limits_{x \to 0} \dfrac{|x|}{x}$
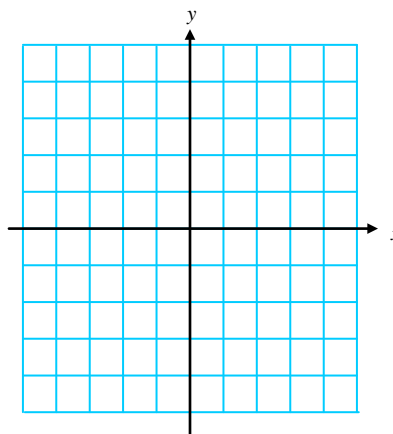
x	−0.5	−0.25	−0.1	−.01	−.001	0	.001	.01	.1	.25	.5
$f(x)$											

(b) $\lim\limits_{x \to 1} \dfrac{1}{(x-1)^2}$

x	0	.5	.9	.99	.999	1	1.001	1.01	1.1	1.5	2
$f(x)$											

(c) $\lim\limits_{x \to 0} \sin\left(\dfrac{1}{x}\right)$ … Try graphing this on your calculator.

First convince yourself that as you move to the right in the chart below x is actually getting closer and closer to 0.

x	$\dfrac{2}{\pi}$	$\dfrac{2}{3\pi}$	$\dfrac{2}{5\pi}$	$\dfrac{2}{7\pi}$	$\dfrac{2}{9\pi}$	$\dfrac{2}{11\pi}$	$\dfrac{2}{13\pi}$	As $x \to 0$
$f(x)$								

Properties of Limits

For many "well – behaved" functions, evaluating the limit can be found by <u>direct substitution</u>. That is,
$$\lim_{x \to c} f(x) = f(c)$$
Such *well – behaved* functions are **continuous at c**. We will study continuity of a function in §2.3.
The following theorems describe limits that can be evaluated by direct substitution.

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.
$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

$$\lim_{x \to c} b = b \qquad\qquad \lim_{x \to c} x = c \qquad\qquad \lim_{x \to c}\left[f(x) \pm g(x) \right] = L \pm K$$

$$\lim_{x \to c}\left[f(x) \cdot g(x) \right] = LK \qquad \lim_{x \to c}\left[b \cdot f(x) \right] = bL \qquad \lim_{x \to c}\left[f(x) \right]^{r/s} = L^{r/s}$$

$$\text{provided } r \text{ and } s \text{ are integers and } s \neq 0$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K} \quad ; \text{provided } K \neq 0$$

You should realize that all the properties in the box above basically say that you can evaluate a limit as x approaches c by plugging c into the equation.

Example: Find each limit

    (a) $\lim\limits_{x \to 1}\left(-x^2 + 1\right)$ (b) $\lim\limits_{x \to 4}\sqrt[3]{x + 4}$

    (c) $\lim\limits_{x \to 3}\dfrac{\sqrt{x+1}}{x - 4}$ (d) $\lim\limits_{x \to 7}\sec\left(\dfrac{\pi x}{6}\right)$

Example: Use the given information to evaluate the limits: $\lim\limits_{x \to c} f(x) = 2$ and $\lim\limits_{x \to c} g(x) = 3$

    (a) $\lim\limits_{x \to c}\left[5g(x) \right]$

    (b) $\lim\limits_{x \to c}\left[f(x) + g(x) \right]$

    (c) $\lim\limits_{x \to c}\left[f(x) g(x) \right]$

    (d) $\lim\limits_{x \to c}\dfrac{f(x)}{g(x)}$

A Strategy for Finding Limits

If a limit cannot be found using direct substitution, then we will use other techniques to evaluate the limit.

♫ : Keep in mind that some functions do not have limits.

If direct substitution yields the <u>meaningless</u> result $\dfrac{0}{0}$, then you ***cannot determine*** the limit in this form.

The expression that yields this result is called an **Indeterminate Form**. … *DO SOMETHING ELSE!*
When you encounter this form, you must rewrite the fraction so that the new <u>denominator</u> does not have 0 as its limit. One way to do this is to *cancel like factors*, and a second way is to *rationalize the <u>numerator</u>*.
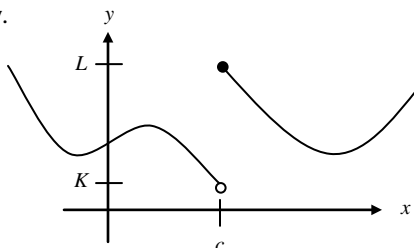

Example: Find the limit: $\displaystyle\lim_{x \to -1} \frac{2x^2 - x - 3}{x + 1}$

Example: Find the limit (if it exists): $\displaystyle\lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x - 3}$

Example: Find the limit (if it exists): $\displaystyle\lim_{x \to 0} \frac{\left[1/(x+4)\right] - (1/4)}{x}$

One – Sided Limits

One of the reasons a limit did not exist is because the function approached a different value from the left than it did from the right. Suppose we have the graph below.

Earlier, we would have said that the limit as x approaches c does not exist because as x approaches c from the left, the function approaches K, and as x approaches c from the right, the function approaches L. However, sometimes we are interested in what the function approaches as x approaches only from the right or left of c. We can say this using the following notation:

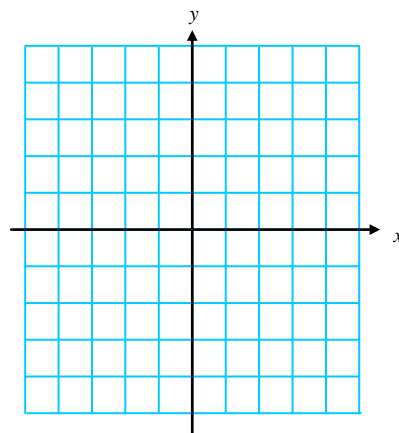$$\lim_{x \to c^+} f(x) = L \text{ ... "the limit of } f(x) \text{ as } x \text{ approaches } c \text{ from the right is } L\text{"}$$

$$\lim_{x \to c^-} f(x) = K \text{ ... "the limit of } f(x) \text{ as } x \text{ approaches } c \text{ from the left is } K\text{"}$$

Thus, we can say that the limit of a function as x approaches any number c exists if and only if the limit as x approaches c from the right is equal to the limit as x approaches c from the left. Using limit notation we have
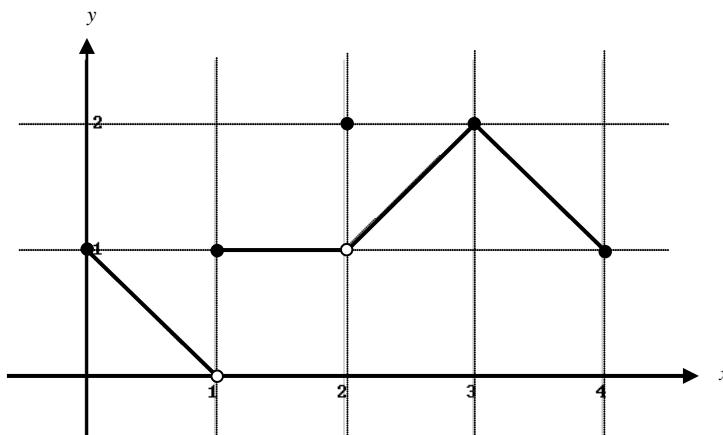
$$\lim_{x \to c} f(x) \text{ exists } \Leftrightarrow \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$$

Example: Let $f(x) = \lfloor x \rfloor$

   a) Graph $f(x)$

   b) Find $\lim_{x \to 2^+} f(x)$.

   c) Find $\lim_{x \to 2^-} f(x)$

Example: Find $\lim_{x \to c^+} f(x)$, $\lim_{x \to c^-} f(x)$, and $\lim_{x \to c} f(x)$ for $c = 0, 1, 2, 3, 4$. If the limit does not exist, explain why.

The Sandwich Theorem (a.k.a. The Squeeze Theorem)

Example: Investigate the $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$ by sketching a graph and making a table.

You must understand that while using a graph and/or a table, we may be able to determine what a limit is, we have not proved it until we algebraically confirm the limit is what we think it is. The proof of the above limit requires the use of the sandwich theorem.
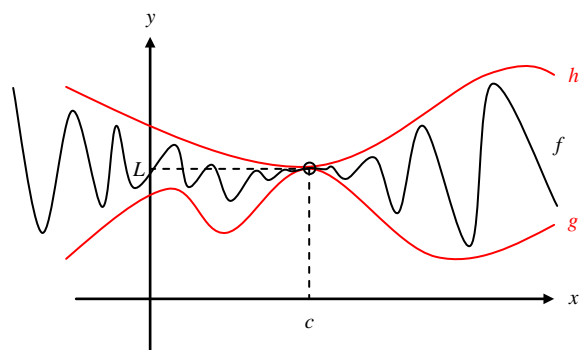

The Sandwich Theorem

If $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some interval about c, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$

then

$$\lim_{x \to c} f(x) = L.$$

In other words, if we "sandwich" the function f between two other functions g and h that both have the same limit as x approaches c, then f is "forced" to have the same limit too.
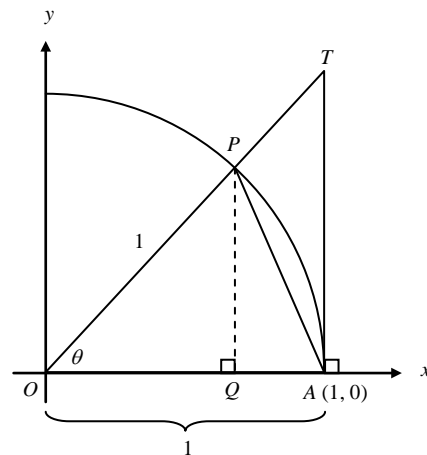
Example: Prove that $\lim\limits_{\theta \to 0} \dfrac{\sin \theta}{\theta} = 1$. To do this, we are going to use the figure below. Admittedly, the toughest part of using the sandwich theorem is finding two functions to use as "bread" ☺.

First, we need to find $\lim\limits_{\theta \to 0^+} \dfrac{\sin \theta}{\theta}$. In order to do this we need to restrict θ so that $0 < \theta < \dfrac{\pi}{2}$. Why are we able to do this?

a) Find the area of $\triangle OAP$.

b) Find the area of sector OAP.

c) Find the area of $\triangle OAT$.

d) Set up an inequality with the three areas from parts *a*, *b*, and *c*.

e) Divide all three parts by $\dfrac{1}{2}\sin \theta$. Why do the inequality signs stay the same?

f) Make the middle term $\dfrac{\sin \theta}{\theta}$. *Hint*: If your middle term doesn't look anything like this, start over! ☺

g) Use the Sandwich Theorem to show that $\lim\limits_{x \to 0^+} \dfrac{\sin \theta}{\theta} = 1$.

h) Show that $f(\theta) = \dfrac{\sin \theta}{\theta}$ is an even function.

i) Since $f(\theta) = \dfrac{\sin \theta}{\theta}$ is an even function, what can you conclude about $\lim\limits_{x \to 0^-} \dfrac{\sin \theta}{\theta}$?

j) Explain why we can conclude that $\lim\limits_{x \to 0} \dfrac{\sin \theta}{\theta} = 1$.

Example: Find $\lim\limits_{\text{☺}\to 0}\dfrac{\sin\text{☺}}{\text{☺}}$

Example: Find $\lim\limits_{x\to 0}\dfrac{\sin 5x}{x}$

Example: Find $\lim\limits_{x\to 0}\dfrac{\sin x}{5x}$

Example: Find $\lim\limits_{x\to 0}\dfrac{\sin 5x}{4x}$

Example: Find $\lim\limits_{x\to 0}\dfrac{\sin x}{5x^2 + x}$

*Example*: Find $\lim\limits_{x\to 0}\dfrac{\tan x}{x}$

Notecards from Section 2.1: How to Evaluate a Limit, Properties of Limits, Definition of a Limit at $x = c$, When Limits fail, and Limits you should know.

2.2 LIMITS INVOLVING INFINITY

We are going to look at two kinds of limits involving infinity. We are interested in determining what happens to a function as x approaches infinity (in both the positive and negative directions), and we are also interested in studying the behavior of a function that approaches infinity (in both the positive and negative directions) as x approaches a given value.

Finite Limits as $x \to \pm\infty$

Example: Investigate $\lim\limits_{x \to \infty} f(x)$ and $\lim\limits_{x \to -\infty} f(x)$ for $f(x) = \dfrac{2x-1}{x+3}$.
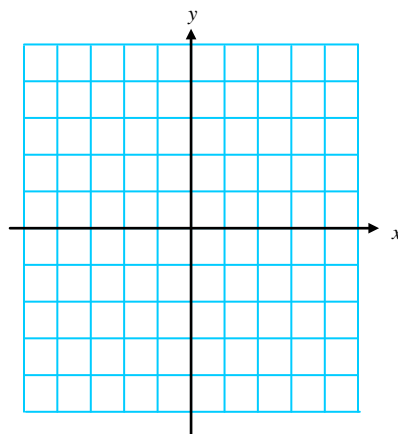
Example: Investigate $\lim\limits_{x \to \infty} f(x)$ and $\lim\limits_{x \to -\infty} f(x)$ for $f(x) = \dfrac{1}{x}$.
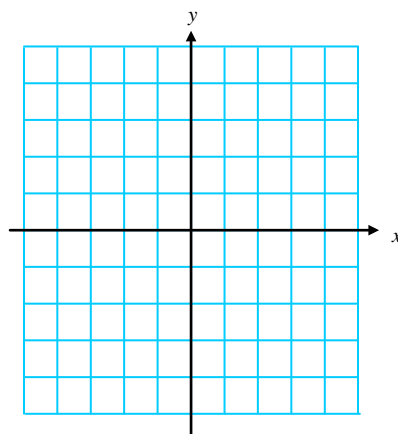
Both of these graphs have horizontal asymptotes. Are you able to determine what they are from the graphs?

Definition: Horizontal Asymptote

The line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim\limits_{x \to \infty} f(x) = b \quad \text{or} \quad \lim\limits_{x \to -\infty} f(x) = b$$

In the last example, we saw that $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$. Using this limit and the properties of limits, we can find the limits of other functions as x approaches infinity.

Example: $\lim\limits_{x \to \infty} \left(5 - \dfrac{2}{x^2} \right) =$

End Behavior Models

Definition: End Behavior Model

For a rational function $\dfrac{ax^m + \cdots}{bx^n + \cdots}$, where m is the degree of the numerator and n is the degree of the denominator. The end

behavior model can be written as $\dfrac{ax^m}{bx^n}$ or $\dfrac{a}{b}x^{m-n}$.

We can use the end behavior models of _rational_ functions to identify any _horizontal asymptotes_ the function.

Example: For each example below do the following:

> _i_) Write the end behavior model.
> _ii_) Evaluate each limit.
> _iii_) Determine whether or not there are any horizontal asymptotes. If so, what is the equation?
> _iv_) Determine whether or not there are any slant (oblique) asymptotes. If so, what is the equation?

a) $\displaystyle\lim_{x \to \infty} \frac{2x+5}{3x^2 - 6x + 1}$

b) $\displaystyle\lim_{x \to -\infty} \frac{2x+5}{3x^2 - 6x + 1}$

c) $\displaystyle\lim_{x \to \infty} \frac{2x^2 - 3x + 5}{x^2 + 1}$

d) $\displaystyle\lim_{x \to -\infty} \frac{2x^2 - 3x + 5}{x^2 + 1}$

e) $\displaystyle\lim_{x \to \infty} \frac{x^4 + x^3 + 9}{3x - 3}$

In all of the examples above, if there was a limit as x approached positive infinity, the limit as x approached negative infinity was the same. Thus, there was one horizontal asymptote, and the function approached this same asymptote in both directions. However, certain functions, especially irrational functions, have more than one horizontal asymptote.

Example: Investigate $\displaystyle\lim_{x \to \infty} \frac{3x - 2}{\sqrt{2x^2 + 1}}$ and $\displaystyle\lim_{x \to -\infty} \frac{3x - 2}{\sqrt{2x^2 + 1}}$

If $f(x) = ax^m + \cdots$ and $g(x) = bx^n + \cdots$ then $\dfrac{f(x)}{g(x)}$ takes on three different forms.

	End Behavior Model	End Behavior	Asymptote
Degrees are equal $m = n$			
Larger degree in denominator $m < n$			
Larger degree in numerator $m > n$			

Example: In the last section we proved that $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$. Investigate $\lim\limits_{x \to \infty} \dfrac{\sin x}{x}$.

It should come as no surprise that the proof of this limit involves the Sandwich Theorem. ☺ See the text if you just can't contain your curiosity. The limit properties that we used when x approaches c are still valid as x approaches infinity.

Example: $\lim\limits_{x \to \infty} \left(\dfrac{2}{x} + 1 \right) \left(\dfrac{5x^2 - 1}{x^2} \right)$

Example: $\lim\limits_{x \to \infty} \dfrac{x \sin x + 2 \sin x}{2x^2}$

Infinite Limits as $x \to a$

A second type of limit involving infinity is to determine the behavior of the function as x approaches a certain value when the function increases or decreases without bound.

Example: Investigate $\lim\limits_{x \to 0^+} \dfrac{1}{x}$ and $\lim\limits_{x \to 0^-} \dfrac{1}{x}$.

In the last example we call the line $x = 0$ a vertical asymptote.

Definition: Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty$$

This occurs whenever there is a value of x that gives you a 0 in the denominator (but not the numerator).

Important ♪ : Infinity is NOT a number, and thus the limit FAILS to exist in both of these cases. If this seems confusing, then use the notation as $x \to a$ (from the right or left), then the function $f(x) \to \pm\infty$.

Example: Find the vertical asymptotes of $f(x)$. Describe the behavior of $f(x)$ to the left <u>and</u> right of each asymptote.

    a) $f(x) = \dfrac{x^2 - 1}{2x + 4}$

    b) $f(x) = \dfrac{1 - x}{2x^2 - 5x - 3}$

    c) $f(x) = \dfrac{x - 2}{3x^2 - 5x - 2}$

Example: Sketch the function that satisfies the stated conditions.

$$\lim_{x \to 1} f(x) = 2 \qquad\qquad \lim_{x \to 5^-} f(x) = \infty$$

$$\lim_{x \to 5^+} f(x) = \infty \qquad\qquad \lim_{x \to \infty} f(x) = -1$$

$$\lim_{x \to -\infty} f(x) = 0 \qquad\qquad \lim_{x \to -2^-} f(x) = \infty$$

$$\lim_{x \to -2^+} f(x) = -\infty$$

Example: Sketch the function that satisfies the stated conditions.

$$\lim_{x \to 2} f(x) = -1 \qquad\qquad \lim_{x \to 4^+} f(x) = -\infty$$

$$\lim_{x \to 4^-} f(x) = \infty \qquad\qquad \lim_{x \to \infty} f(x) = \infty$$

$$\lim_{x \to -\infty} f(x) = 2$$

Example: Answer the following questions:

    a) How do you find horizontal asymptotes?

    b) How do you find vertical asymptotes?

    c) How do you find oblique (slanted) asymptotes?

Notecards from Section 2.2: Definition of a horizontal asymptote, Definition of a vertical asymptote, End Behavior, End Behavior Models, Oblique (Slant) Asympototes
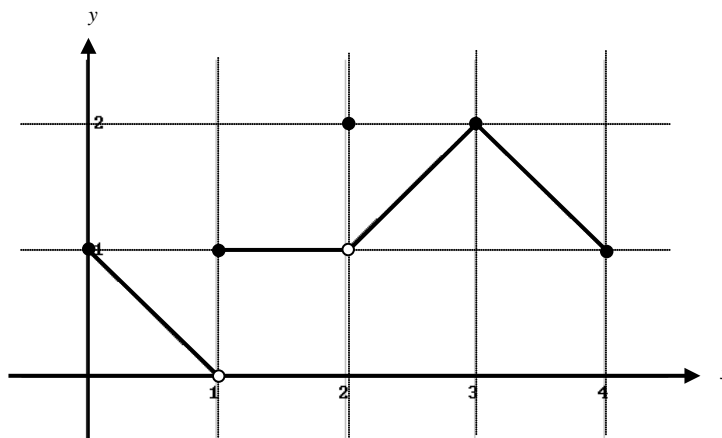
2.3 CONTINUITY

On page 29 of your notes from §2.1 we referred to "well behaved" functions. "Well behaved" functions allowed us to find the limit by direct substitution. "Well behaved" functions turn out to be continuous functions. In this section we will discuss continuity at a point, continuity on an interval, and the different types of discontinuities.

In non – technical terms, a function is continuous if you can draw the function "without ever lifting your pencil". The following graphs demonstrate three types of discontinuous graphs.
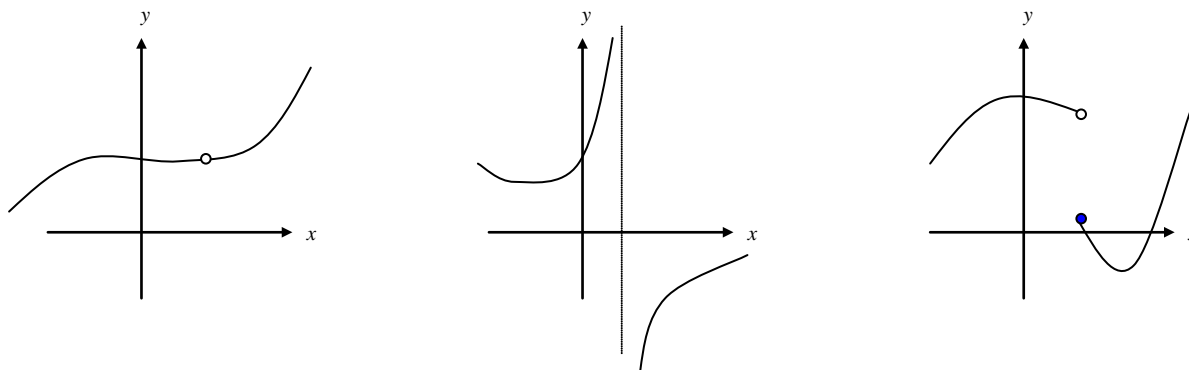
Let's go back to the example we used in §2.1 when we discussed one – sided limits.

Example: Find the points (intervals) at which the function above is continuous, and the points at which it is discontinuous.

Example: For each point of discontinuity, c, find $f(c)$, $\lim\limits_{x \to c^+} f(x)$, $\lim\limits_{x \to c^-} f(x)$, and $\lim\limits_{x \to c} f(x)$ if they exist.

Definition: *Continuity at a Point*

Interior Point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if
$$\lim_{x \to c} f(x) = f(c)$$

♪: This last statement implies that $\lim_{x \to c} f(x)$ exists. This limit only exists if the limit from the left and right of c are equal!
It also implies that the function value at c … $f(c)$ exists.

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if
$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{or} \qquad \lim_{x \to b^-} f(x) = f(b), \qquad \text{respectively.}$$

Discontinuities: Removable versus Non-Removable

To say a function is discontinuous is not sufficient. We would like to know what type of discontinuity exists. If the function is not continuous, but I <u>could make it continuous by appropriately defining or redefining f(c),</u> then we say that f has a **removable discontinuity**. Otherwise, we say f has a **non-removable discontinuity**.

Once again, informally we say that f has a <u>removable discontinuity</u> if there is a "<u>hole</u>" in the function, but f has a non-removable discontinuity if there is a "jump" or a vertical asymptote.

All polynomials are continuous. For rational functions, we try to algebraically "remove" the discontinuity by canceling factors found in both the denominator and the numerator if possible.

Example: Which (if any) of the three graphs at the top of the other side of the paper have a removable discontinuity?

Example: Discuss the continuity of each function

(a) $f(x) = \dfrac{1}{x-1}$

(b) $g(x) = \dfrac{2x^2 + x - 6}{x + 2}$

(c) $h(x) = \begin{cases} -2x + 3 & ; x < 1 \\ x^2 & ; x \geq 1 \end{cases}$

Example: Determine the value of k such that the function is continuous on the entire real line.

$$g(x) = \begin{cases} x^2 + 7 & \text{if } x \geq 1 \\ x + k & \text{if } x < 1 \end{cases}$$

Example: Determine the value of k in order to make the function continuous for all real numbers.

$$h(x) = \begin{cases} \dfrac{x^4 - 1}{x - 1} & \text{if } x \neq 1 \\ k & \text{if } x = 1 \end{cases}$$

Properties of Continuity

Since continuity is defined using limits, the properties of limits carry over into continuity.

Properties of Continuity

If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c.

1. Constant multiple: bf 2. Sum and difference: $f \pm g$

3. Product: fg

4. Quotient: $\dfrac{f}{g}$; $g(c) \neq 0$

The Intermediate Value Theorem (IVT)

If f is continuous on the closed interval [a, b] then f takes on every value between $f(a)$ and $f(b)$. Suppose k is any number between $f(a)$ and $f(b)$, then there is at least one number c in [a, b] such that $f(c) = k$.

♪: The Intermediate value theorem tells you that at least one c exists, but it does not give you a method for finding c. This theorem is an example of an *existence theorem*.

Example: In the Intermediate Value Theorem, which axis is k on? What about c?

Example: Consider the function f below.
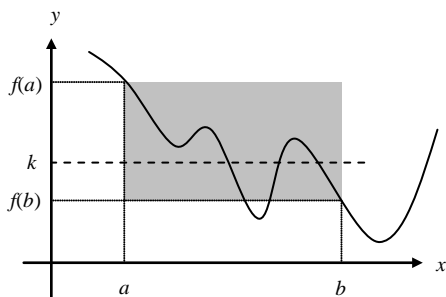
➤ Is f continuous on [a, b] ?

➤ Is $f(b) < k < f(a)$?

➤ In this example, if $a < c < b$, then there are _____ c's such that $f(c) = k$.

➤ Label the c's on the graph as $c_1, c_2, \ldots$

Example: Is there any real number exactly 2 more than its cube? Give any such values accurate to 3 decimal places.

Example: Let $f(x) = \dfrac{x^2 + x}{x - 1}$. Verify that the Intermediate Value Theorem applies to the interval $\left[\dfrac{5}{2}, 4\right]$ and find the value of c guaranteed by the theorem if $f(c) = 6$.

Notecards from Section 2.3: Definition of continuity at $x = c$, Types of Discontinuities, Intermediate Value Theorem

## 2.4 RATES OF CHANGE AND TANGENT LINES

Average Rates of Change

Example: Remember this example? … Wile E. Coyote, once again trying to catch the Road Runner, waits for the nastily speedy bird atop a 900 foot cliff. With his Acme Rocket Pac strapped to his back, Wile E. is poised to leap from the cliff, fire up his rocket pack, and finally partake of a juicy road runner roast. Seconds later, the Road Runner zips by and Wile E. leaps from the cliff. Alas, as always, the rocket malfunctions and fails to fire, sending poor Wile E. plummeting to the road below disappearing into a cloud of dust.

Let's look at this problem from a graphical perspective. The equation that models Wile E.'s height at any time *t* is given by

$$s(t) = -16t^2 + 900$$

A graph of this equation is shown below.
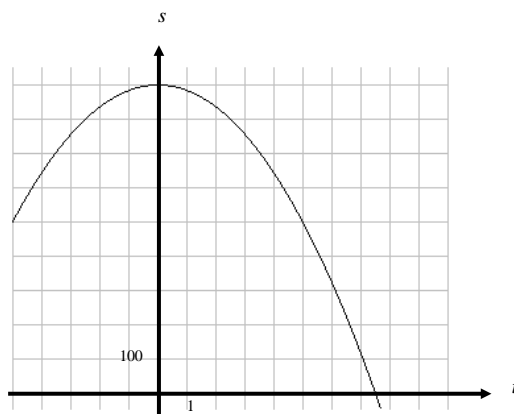
Example: Find the points on the graph that correspond to Wile E's position at *t* = 0 and *t* = 5 seconds.

Example: Draw the line that passes through these points, and find the slope. What does this value mean?

The line you drew on the graph can be called a *secant line*. <u>A secant line is a line through any two points on a curve</u>. Just like we did in this example, we can always think of the **average rate of change** as the slope of the secant line. To find the slope of the secant line above we divided the total change in *s* by the total change in *t*. To find the average rate of change in the position (a.k.a. velocity) we found the total change in position divided by the total change in time.

We are able to find Wile E.'s average velocity for any period of time following the same procedure as above. Do you remember the problem we had finding the velocity of poor Wile E. Coyote at an exact moment in time? If we wanted to find the velocity of Wile E. Coyote at *exactly* 5 seconds, we tried to determine the average velocity using values of *t* that were closer and closer to *t* = 5.

Example: Find Wile E.'s average velocity (rate of change) from *t* = 4 to *t* = 5 seconds. Graphically show this above.

Example: Find Wile E.'s average velocity (rate of change) from *t* = 4.5 to *t* = 5 seconds. Graphically show this above.

Example: Find Wile E.'s average velocity (rate of change) from *t* = 4.9 to *t* = 5 seconds. Graphically show this above.

*Example*: What do you think would be the graphical interpretation of the velocity at exactly 5 seconds?

Tangent to a Curve

For a circle, the tangent line at a point P is the line that is perpendicular to the radial line at point P.
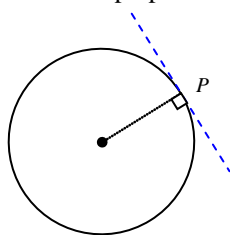
For a general curve, however, the problem of defining a tangent line is more difficult.

Example: Think of a definition of a tangent line. Using each picture (with tangent line shown) does your definition hold?
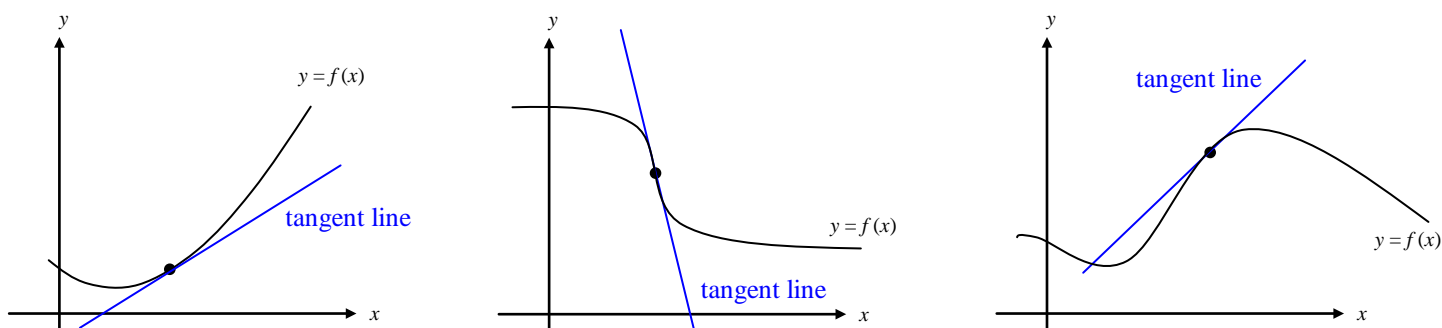

Finding the equation of the tangent line simply boils down to finding the <u>*slope*</u> of the tangent line. (Since we already have a <u>point</u> of tangency, if we knew the slope we would be able to write the equation of a line.)

We can *approximate* the slope of the tangent line using a **secant line**.

If $P\,(a, f(a))$ is the point of tangency we are concerned with, then we can pick an arbitrary point Q on the graph and estimate the tangent line at P using the slope of the secant line through P and Q.

Example: What is the slope of the secant line?
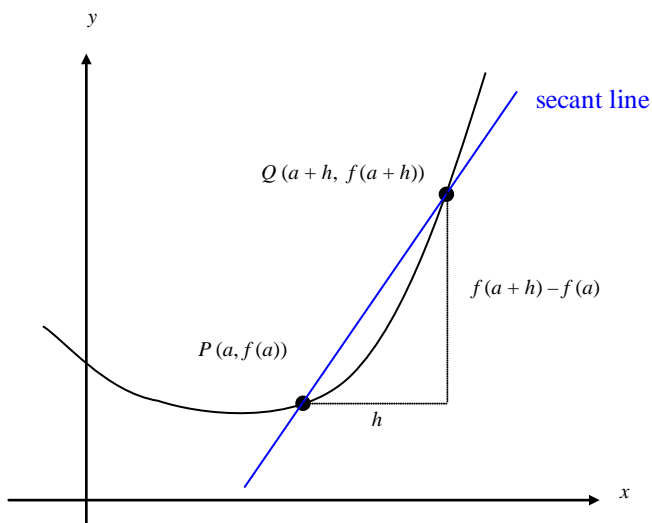
The beauty of this procedure is that you can obtain more and more accurate approximations to the slope of the tangent line by choosing points closer and closer to the point of tangency. How do we get closer and closer to the point of tangency? Draw at least 3 more secant lines, using a point closer to P each time.

As $h \to 0$, the slope of the secant line approaches the slope of the tangent line.

Slope of a Curve

The slope of a line is always constant. The slope of a curve is constantly changing. Think of a curve as a roller coaster that you are riding. If for some reason the "track" were to just disappear, you would go flying off in the direction that you were traveling at that last instant before the track disappeared. The direction that you flew off to would be the slope of the curve at that point.

Using the slope of the secant line from the last example, we have the following definition.

Slope of a Curve at a Point

The **slope of the curve** $y = f(x)$ at the point $P(a, f(a))$ is the number
$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$
provided the limit exists.

♫: The expression $\dfrac{f(a+h) - f(a)}{h}$ is called a **difference quotient**.

Your goal is to SIMPLIFY the difference quotient, THEN evaluate the limit as h approaches 0.

The difference quotient is simplified when you have cancelled the h in the denominator in an algebraically correct way.

The **tangent line to the curve** at P is the line through P with this slope.

Example: According to this definition, when would the tangent line not exist?

**To cover the possibility of a vertical tangent line, we can use the following definition.

If f is continuous at a and
$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \pm\infty$$
the vertical line, $x = a$ is a **vertical tangent line** to the graph of f.

Example: What types of graphs would have vertical tangent lines?

Before we use this definition, be sure to become comfortable with the notation $f(a + h)$.

Example: If $f(x) = \dfrac{1}{x}$, what is $f(a)$? ... $f(a+h)$?

Example: If $f(x) = x^2 - 4x$, what is $f(a)$? ... $f(a+h)$?

Example: If $f(x) = \sqrt{4x+1}$, what is $f(a)$? ... $f(a+h)$?

Example: Let $f(x) = -16x^2 + 900$. Find the slope of the curve at $x = 5$.

♪: This would be the **instantaneous velocity** of Wile E. Coyote at exactly 5 seconds!

Normal Line

The **normal line** to a curve at a point is the line perpendicular to the tangent at that point.

Example: Let $f(x) = \dfrac{1}{x+1}$.

    a) Find the slope of the curve at $x = a$.

    b) Find the slope of the curve at $x = 2$.

    c) Write the equation of the tangent line to the curve at $x = 2$.

    d) Write the equation of the normal line to the curve at $x = 2$.

Example: Let $f(x) = x^3$.

    a) Find the slope of the curve at $x = a$.

    b) When does the slope equal 12?

    c) Write the equation of the tangent line to the curve at $x = 4$.

    d) Write the equation of the normal line to the curve at $x = 4$.

Notecards from Section 2.4: Definition of Average Rate of Change, Definition of Instantaneous Rate of Change.